Principle Series Representation Part II.

We have constructed: a space of functions on F

$$\mathcal{B}_{\mu_1,\mu_2} \xrightarrow[\simeq]{} \mathcal{F}_{\mu_1\mu_2^{-1}} \xrightarrow{\Phi_\varphi} \mathcal{K}_{\mu_1,\mu_2}$$

$$\varphi \longmapsto x \mapsto \mu_2(x)|x|^{\frac{1}{2}}\hat{\Phi}_\varphi(x)$$

- $\mathcal{K}_{\mu_1,\mu_2} = \left\{ \xi \in C^\infty(F^\times) \;\middle|\; \begin{array}{l} x \text{ large}, \ \xi \text{ vanishes} \\ x \text{ near } 0, \ \xi(x) = \begin{cases} |x|^{\frac{1}{2}}(a\mu_1(x) + b\mu_2(x)), & \mu_1\mu_2^{-1} \neq \mathbb{1}, |\,|^{-1} \\ |x|^{\frac{1}{2}}(a\mu_2(x)v(x) + b\mu_2(x)), & \mu_1\mu_2^{-1} = \mathbb{1} \\ b|x|^{\frac{1}{2}}\mu_2(x), & \mu_1\mu_2^{-1} = |\,|^{-1}. \end{cases} \end{array} \right\}$

- Map is isom if $\mu_1\mu_2^{-1} \neq |\,|^{-1}$.
  Kernel is span of $\mu_1 \circ \det |\det|^{\frac{1}{2}}$, if $\mu_1\mu_2^{-1} = |\,|^{-1}$. (Kernel of $\mathcal{F}_{\mu_1\mu_2^{-1}} \to \mathcal{K}_{\mu_1,\mu_2}$ is const funs)

- If we transport the repn. of G on $(\rho_{\mu_1,\mu_2}, \mathcal{B}_{\mu_1,\mu_2})$ to $\mathcal{K}_{\mu_1,\mu_2}$ and call it κ_{μ_1,μ_2}, then $\kappa_{\mu_1,\mu_2}\begin{pmatrix} a & b \\ & 1 \end{pmatrix} \xi(x) = \tau_F(bx)\,\xi(ax)$.

Thm 6

① If $\mu_1\mu_2^{-1} \neq |\ |,\ |\ |^{-1}$, ρ_{μ_1,μ_2} irred.

② If $\mu_1\mu_2^{-1} = |\ |^{-1}$, B_{μ_1,μ_2} contains a 1-dim subrepn. generated by $\mu_1\circ\det\,|\det|^{\frac{1}{2}}$. The quotient space is irred.

③ If $\mu_1\mu_2^{-1} = |\ |$, B_{μ_1,μ_2} contains a irred. subrepn. of codim 1, consisting of functions φ s.t.

$$\int_{B\backslash G} \varphi(g)\,\mu_1^{-1}(\det g)\,|\det g|^{\frac{1}{2}}\,d\bar{g} = 0$$

$$= \int_F \varphi\left(w^{-1}\begin{pmatrix}1 & x\\ & 1\end{pmatrix}\right) dx$$

using additive notation for characters

($B_{\mu_1,\mu_2} \times B_{-\mu_1,-\mu_2} \longrightarrow \mathbb{C}$ non-deg pairing.)

$\varphi \in B_{\mu_1,\mu_2}$ orthog to $\longrightarrow$ $\mu_1^{-1}\circ\det\,|\det|^{\frac{1}{2}} \in B_{-\mu_1,-\mu_2}$

Pf : Step 1

$$\text{inv. subsp} \subset K_{\mu_1,\mu_2}$$

$$\xi \circlearrowright$$

$$\xi - K_{\mu_1,\mu_2}\begin{pmatrix}1 & b\\ & 1\end{pmatrix}\xi$$

$$x \mapsto \xi(x) - \tau_F(bx)\xi(x) \in \mathcal{S}(F^\times)$$

Any non-zero inv. subsp. of K_{μ_1,μ_2} contains a non-zero fun in $\mathcal{S}(F^\times)$.

$\Rightarrow$ - ——— ——— — ——— ——— - — - all of $\mathcal{S}(F^\times)$

b/c $\mathcal{S}(F^\times) \circlearrowleft$ mirabolic $\left\{\begin{pmatrix}a & b\\ & 1\end{pmatrix}\right\}$ is irred.

Step 2

assumed to be nonzero.
in other words, $V \neq \mathbb{C}\mu_1 \circ \det |\det|^{\frac{1}{2}}$ if $\mu_1\mu_2^{-1} = |\cdot|^{-1}$.
and $V \neq 0$.

$$\begin{array}{cccccc} & \mathcal{B}_{\mu_1,\mu_2} & \longrightarrow & \mathcal{K}_{\mu_1,\mu_2} & & \\ & \cup & & \cup & & \\ \text{subrepn.} & V & \longrightarrow & \mathcal{K}_V & \supset & \mathcal{S}(F^\times) \\ & \Downarrow & & & & \Downarrow \end{array}$$

$$\varphi - \rho_{\mu_1,\mu_2}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\varphi \xleftarrow{\text{implies}} \xi - \mathcal{K}_{\mu_1,\mu_2}\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}\xi \quad \forall \xi \in \mathcal{K}_{\mu_1,\mu_2},$$

$\forall \varphi \in \mathcal{B}_{\mu_1,\mu_2}$

Non-deg. pairing: Exclude the V = 1-dim case.

$$\begin{array}{ccccc} \mathcal{B}_{\mu_1,\mu_2} & \times & \mathcal{B}_{-\mu_1,-\mu_2} & \longrightarrow & \mathbb{C} \\ \cup & & \cup & & \\ V & & V^\perp & & \end{array}$$

Every $\psi \in V$ is fixed by $\begin{pmatrix} 1 & b \\ & 1 \end{pmatrix}$ $\forall b$.

$$\psi\Big(\underbrace{\begin{pmatrix} * & * \\ & * \end{pmatrix} w^{-1} \begin{pmatrix} 1 & * \\ & 1 \end{pmatrix}}_{\substack{\text{open cell} \\ \text{dense in } G_1}}\Big) \text{ is determined by value } \psi(w^{-1}).$$

$\Rightarrow \psi$ ______ ______ __ ______ .

$\Rightarrow \dim V^\perp \leq 1.$ $\quad$ $\operatorname{codim} V \leq 1.$ (some reasoning on next page)

Such a ψ $(\neq 0)$ exists only when $\mu_1 \mu_2^{-1} = |\ |$. In this case $\psi = \mu_1^{-1} \circ \det \, |\det|^{\frac{1}{2}}$ up to const.

Pf: $\psi\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \psi\left(\begin{pmatrix} c^{-1}\det g & * \\ & c \end{pmatrix} w^{-1} \begin{pmatrix} 1 & c^{-1}d \\ & 1 \end{pmatrix}\right)$

$c \neq 0$. $\quad = \mu_1^{-1}(c^{-1}\det g)\, \mu_2^{-1}(c) \left| c^{-2}\det g \right|^{\frac{1}{2}} \psi(w^{-1})$.

c near 0 $\quad = \mu_1(c)\mu_2^{-1}(c)\, \mu_1^{-1}(\det g)\, |c|^{-1} |\det g|^{\frac{1}{2}} \psi(w^{-1})$

$\psi\begin{pmatrix} a & b \\ & d \end{pmatrix} = \psi\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ $\quad$ //

$\qquad\qquad\qquad\qquad$ c small

$\|$

$\mu_1^{-1}(a)\, \mu_2^{-1}(d) \left| a d^{-1} \right|^{\frac{1}{2}} \psi(1) = \mu_1(d)\mu_2^{-1}(d)\, |d|^{-1} \mu_1^{-1}(\det g)\, |\det g|^{\frac{1}{2}} \psi(1)$

this is possible only when $\mu_1 \mu_2^{-1} |\ |^{-1} = \mathbb{1}$.

i.e. $\mu_1 \mu_2^{-1} = |\ |$.

Get Jordan-Hölder series:

$\mathcal{B}_{\mu_1,\mu_2}$ irred if $\mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}$.

$\mathcal{B}_{\mu_1,\mu_2} \supset$ 1-dim $\mu_1 \circ \det |\det|^{\frac{1}{2}}$ if $\mu_1\mu_2^{-1} = |\ |^{-1}$.

the quotient

$\mathcal{B}_{\mu_1,\mu_2} \supset$ codim 1 irred. if $\mu_1\mu_2^{-1} = |\ |$

dual to $\mu_1^{-1} \circ \det |\det|^{\frac{1}{2}} \subset \mathcal{B}_{-\mu_1,-\mu_2}$. $\square$

called <u>special repn.</u> σ_{μ_1,μ_2} $\sigma_{\mu_1,\mu_1|\ |}$ $\sigma_{\mu_1,\mu_1|\ |^{-1}}$. (in Jacquet-Langlands notation)

<u>principal series repn</u>

Remarks on Kirillov model.

$f, f_1, f_2 \in \mathcal{S}(F)$.

$\mu_1\mu_2^{-1} \neq |\ |, |\ |^{-1}$ $\quad \pi = \rho_{\mu_1,\mu_2}$ $\quad K(\pi) = \mathcal{K}_{\mu_1,\mu_2} \overset{\text{codim } 2}{\supset} \mathcal{S}(F^\times)$

$$\begin{cases} |x|^{\frac{1}{2}}\mu_1(x) f_1(x) + |x|^{\frac{1}{2}}\mu_2(x) f_2(x), & \mu_1\mu_2^{-1} \neq \mathbb{1} \\ |x|^{\frac{1}{2}}\mu_2(x) f_1(x) + |x|^{\frac{1}{2}}\mu_2(x) v(x) f_2(x), & \mu_1\mu_2^{-1} = \mathbb{1} \end{cases}$$

$\mu_1\mu_2^{-1} = |\ |^{-1}$ $\quad \pi =$ special repn. $\quad K(\pi) = \mathcal{K}_{\mu_1,\mu_2} \overset{\text{codim } 1}{\supset} \mathcal{S}(F^\times)$ $\quad |x|^{\frac{1}{2}}\mu_2(x) f(x)$

$\mu_1\mu_2^{-1} = |\ |$ $\quad \pi =$ special repn. $\quad K(\pi) \overset{\text{codim } 1}{\subset} \mathcal{K}_{\mu_1,\mu_2}$ $\quad |x|^{\frac{1}{2}}\mu_1(x) f(x)$

$\quad\quad K(\pi) \overset{\text{codim } 1}{\supset} \mathcal{S}(F^\times)$

Thm 7

$$\rho_{\mu_1,\mu_2} \simeq \rho_{\mu_2,\mu_1} \qquad \mu_1\mu_2^{-1} \neq |\,|, |\,|^{-1}.$$

$$\sigma_{\mu_1,\mu_2} \simeq \sigma_{\mu_2,\mu_1} \qquad \mu_1\mu_2^{-1} = |\,|, |\,|^{-1}.$$

No other equivalence.